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Cross-Diffusion-Driven Positive Solutions in a Predator-Prey Model with Prey Refuge and Crowley-Martin Functional Response

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Abstract: This paper investigates a modified Leslie-Gower predator-prey model with prey refuge and a Crowley-Martin type functional response under Neumann boundary conditions. Based on the Routh-Hurwitz criterion, we derive the existence and stability conditions of the unique positive constant equilibrium for the diffusion-free model and the diffusion model. Moreover, we study Turing instability of the unique positive constant equilibrium for the cross-diffusion model. Then, using Young's inequality, Poincaré's inequality and Leray-Schauder degree theory, we obtain the nonexistence and existence of nonconstant positive equilibria for the diffusion model and the cross-diffusion model, respectively. Finally, several numerical simulations are provided to illustrate the stability of positive constant equilibria, Turing instability, the existence of nonconstant equilibria and the role of the prey refuge. The results indicate that cross-diffusion is an important factor driving the emergence of nonconstant positive solutions (including equilibria) in the model; that is, cross-diffusion can lead to a non-uniform coexistence pattern of predator and prey.

Key words: predator-prey model; cross-diffusion; Turing instability; nonconstant positive equilibrium; numerical simulation

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0 Introduction

Biological movements, such as diffusion, aggregation, and their subtle balance, are key features in characterizing population dynamics. The dispersal traits of a species are not limited to movement from *high-density* to *low-density regions* (commonly referred to as *diffusion*), but also include intra- and inter-specific pressures that significantly affect population distribution, which mathematically appear as *cross-diffusion*. For further details, we refer the reader to the paper by Okubo and Levin[1]. As shown by Shigesada *et al*[2], Lou and Ni[3], and Li *et al*[4], although diffusion alone fails to produce spatially non-uniform coexistence patterns in two-species competition models under certain conditions, the inclusion of cross-diffusion makes this possible, thereby breaking the *competitive exclusion principle*. This concept was later adopted by many authors in the study of predator-prey models. Peng *et al*[5] explained the significance of cross-diffusion coefficients in a class of

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predator-prey models and discussed the existence of non-constant positive solutions. Kuto *et al*[6] studied the existence of nonconstant positive solutions in a predator-prey model with cross-diffusion. Lou *et al*[7] investigated the global existence of solutions in two spatial dimensions when one of the cross-diffusion coefficients is zero. Furthermore, Zhang *et al*[8] showed that cross-diffusion can affect the formation and disappearance of Turing instability in reaction-diffusion systems.

The predator-prey model, which describes the interspecific interaction wherein one population serves as a food resource for another to ensure its survival, is a central focus in biomathematical research and holds significant research value. In 1960, Leslie and Gower proposed the Leslie-Gower predation model in the form of ordinary differential equations[9], emphasizing how prey population levels influence predator numbers. Subsequent studies revealed that predator growth was affected when the balance between the predator's environmental carrying capacity and the prey population size reached a specific threshold. This finding led to the introduction of a modified Leslie-Gower predation model[10-16].

To better capture real-world predator-prey interactions, various functional responses have been proposed by biologists and mathematicians. The Crowley-Martin (C-M) type functional response[17] is given by $f(u,v)=\dfrac{u}{1+\beta u+\alpha v+\alpha\beta uv}$. Compared with the Beddington-DeAngelis (B-D) type $f(u,v)=\dfrac{u}{1+\beta u+\alpha v}$, the C-M response incorporates an additional term $\alpha\beta uv$ in the denominator, reflecting mutual interference among predators. Here, the parameters α and β represent the handling time of the predator per prey and the intensity of interference between predators, respectively. This functional response has been incorporated into different predator-prey frameworks. Ali and Jazar[18] studied a modified Leslie-Gower predator-prey model with a Crowley-Martin type functional response:

$$\begin{cases}\dfrac{\mathrm{d}u}{\mathrm{d}t}=au\left(1-\dfrac{u}{K}\right)-\dfrac{buv}{(1+\alpha u)(1+\beta v)},\\[2ex]\dfrac{\mathrm{d}v}{\mathrm{d}t}=v\left(r-\dfrac{cv}{u+k}\right),\end{cases}\tag{1}$$

where u and v represent the population densities of prey and predator, respectively. All parameters a, b, c, k, r, α, β, K are positive constants. a and K are the birth rate and the carrying capacity of prey population u, respectively. r and b are the intrinsic growth rate and the capture rate of predator, respectively. k is the extent to which the environment provides protection to the predator (or the half-saturation constant in the predator's growth). c is the coefficient of intra-predator competition. α measures the magnitude of interference among prey. β denotes the intensity of interference between predators. Ali and Jazar[18] established conditions for boundedness, permanence, and the existence of a unique positive equilibrium for model (1). They also studied local stability, Hopf bifurcation leading to periodic solutions, and provided criteria for global stability of both the positive equilibrium and the prey-extinction equilibrium using Lyapunov functions and Dulac's criterion. Meng *et al*[19] further incorporated stage structure for the prey into a Crowley-Martin type predator-prey model, and established conditions for local stability, persistence, and global stability of the positive equilibrium. More recently, Liao *et al*[20] applied the Crowley-Martin functional response to a phytoplankton-zooplankton system with a gestation delay in the zooplankton population. They analyzed permanence, local and global stability, and found that although the time delay does not alter the stability of the positive equilibrium, it can induce Hopf bifurcation and periodic oscillations when the delay exceeds a threshold. Using normal form theory and center manifold reduction, they also determined the direction and stability of the bifurcating periodic solutions. Other related studies include Ref. [21] and Ref. [22], which also examine predator-prey dynamics with Crowley-Martin functional response. Despite their different biological backgrounds, these studies all addressed issues such as persistence, stability of equilibria, and Hopf bifurcation.

In addition to the form of the functional response, another important factor in ecology is the presence of refuges. In ecology, a prey refuge is a spatial structure or habitat that reduces predation risk by providing protection to prey. The addition of a refuge typically increases prey density and decreases predator density. Following this line of research, Yue[23] investigated a modified Leslie-Gower predator-prey model incorporating a constant prey refuge and a Holling-type II functional response:

$$\begin{cases}\dfrac{\mathrm{d}u}{\mathrm{d}t}=u\left(a-u-\dfrac{b(1-m)v}{(1-m)u+\beta}\right),\\[2ex]\dfrac{\mathrm{d}v}{\mathrm{d}t}=v\left(r-\dfrac{cv}{(1-m)u+k}\right),\end{cases}\tag{2}$$

where $m\in[0,1)$ is the proportion of the prey population pro-

tected by the refuge ($m=0$ corresponding to no refuge). This study focused on global attractivity of the positive equilibrium, the impact of a constant prey refuge, and the improvement of existing stability conditions.

Model (1) describes a predator-prey system with Crowley-Martin functional response but without prey refuge. Model (2) includes a constant prey refuge but adopts a simpler Holling-type II functional response. However, studies that combine the Crowley-Martin functional response with a prey refuge in a modified Leslie-Gower framework are relatively limited. Given that both predator interference and prey refuge play crucial roles in stabilizing predator-prey interactions, it is of both theoretical and ecological interest to explore their combined effects. Therefore, in this paper, we investigate the following predator-prey model incorporating both the Crowley-Martin functional response and a constant prey refuge:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = au\left(1-\dfrac{u}{K}\right) - \dfrac{b(1-m)uv}{\left(1+\alpha(1-m)u\right)\left(1+\beta v\right)}, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = v\left(r - \dfrac{cv}{(1-m)u+k}\right). \end{cases}$$

To simplify the model and reduce the number of parameters, we rescale the prey variable by setting $\bar{u}=au/K$ and define the new parameters $\bar{\alpha}=(K/a)\alpha$, $\bar{c}=(a/K)c$, $\bar{k}=(a/K)k$. Dropping the bars for simplicity, we obtain the following system:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = u\left(a-u-\dfrac{b(1-m)v}{\left(1+\alpha(1-m)u\right)\left(1+\beta v\right)}\right), \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = v\left(r-\dfrac{cv}{(1-m)u+k}\right). \end{cases} \tag{3}$$

To incorporate spatial effects, reaction-diffusion models have been widely employed. Yin *et al*[24] extended the modified Leslie-Gower predator-prey model with Crowley-Martin functional response to a reaction-diffusion system under homogeneous Neumann boundary conditions, investigating the global existence of positive solutions, dissipation, persistence, stability of constant equilibria, Hopf bifurcation, and the existence and nonexistence of nonconstant positive steady states. Similar diffusive models with Crowley-Martin functional response have also been studied under Dirichlet or mixed boundary conditions, focusing on persistence, global attractors, and diffusion-driven instability[25-26].

Motivated by the above studies, we propose the following reaction-diffusion system incorporating prey refuge and Crowley-Martin functional response:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1\Delta u + u\left(a-u-\dfrac{b(1-m)v}{(1+\alpha(1-m)u)(1+\beta v)}\right), \\ \qquad\qquad x\in\Omega, t>0, \\ \dfrac{\partial v}{\partial t} = d_2\Delta v + v\left(r-\dfrac{cv}{(1-m)u+k}\right), \quad x\in\Omega, t>0, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, \qquad x\in\partial\Omega, t>0, \\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), \quad x\in\overline{\Omega}. \end{cases} \tag{4}$$

where d_1 and d_2 are the diffusion coefficients of prey and predator, respectively, $\Omega\subset\mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, and $\partial/\partial n$ denotes the outward normal derivative on $\partial\Omega$, which ensures that no individuals cross the boundary.

To further investigate the impact of cross-diffusion, we extend model (4) by incorporating cross-diffusion terms. The resulting cross-diffusion system is given by:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1\Delta u + d_{12}\Delta(uv) \\ +u\left(a-u-\dfrac{b(1-m)v}{(1+\alpha(1-m)u)(1+\beta v)}\right), \quad x\in\Omega,\ t>0, \\ \dfrac{\partial v}{\partial t} = d_2\Delta v + d_{21}\Delta(uv) + v\left(r-\dfrac{cv}{(1-m)u+k}\right), \\ \qquad\qquad x\in\Omega, t>0, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, \qquad x\in\partial\Omega, t>0, \\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), \quad x\in\overline{\Omega}, \end{cases} \tag{5}$$

where d_{12} and d_{21} are cross-diffusion coefficients. These two cross-diffusion terms represent the movement of prey away from predators to escape predation, and the aggregation of prey individuals into large groups to protect themselves from predator attacks[5], respectively.

The remainder of this paper is organized as follows. In Section 1, we analyze the local asymptotic stability of the constant equilibria for systems (3) - (5) and derive conditions for Turing instability induced by cross-diffusion. In Section 2, using Young's inequality and Poincaré's inequality, we prove the nonexistence of nonconstant positive equilibria for system (4), while applying Leray-Schauder degree theory, we establish the existence for system (5). Section 3 presents numerical simulations to verify the theoretical findings, and Section 4 summarizes the main conclusions of this paper.

1 Turing Instability

This section discusses the stability of non-negative

constant equilibria for system (3). For systems (4) and (5), we analyze the stability and Turing instability of the unique positive constant equilibrium. To determine the unique positive constant equilibrium, we recall a classical result on the positive roots of a polynomial.

Lemma 1[27] (Descartes' Rule of Signs)

Suppose

$$f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n$$

is a polynomial with real coefficients, where a_0, a_1, $\cdots$, a_{n-1}, a_n are known constants, and the number of positive roots of the equation $f(x)=0$ is equal to the number of variations of the coefficient list a_0, a_1, $\cdots$, a_{n-1}, a_n minus a non-negative even number.

A straightforward calculation shows that systems (3) - (5) possess the following non-negative constant equilibria:

(i) the trivial equilibrium $(0, 0)$;

(ii) the semi-trivial equilibria $(a, 0)$ and $(0, rk/c)$;

(iii) the positive constant equilibrium $\tilde{\boldsymbol{w}}=(\tilde{u}, \tilde{v})$, where $(\tilde{u}, \tilde{v})$ satisfies the following equations:

$$\begin{cases} a-\tilde{u}-\dfrac{b(1-m)\tilde{v}}{(1+\alpha(1-m)\tilde{u})(1+\beta)v}=0, \\ r-\dfrac{c\tilde{v}}{(1-m)\tilde{u}+k}=0. \end{cases} \tag{6}$$

Eliminating $\tilde{v}$ yields that $\tilde{u}$ satisfies the following cubic equation:

$$\hat{A}\tilde{u}^3+\hat{B}\tilde{u}^2+\hat{C}\tilde{u}+\hat{D}=0, \tag{7}$$

where

$$\hat{A}=\alpha\beta r(1-m)^2,$$

$$\hat{B}=\beta r(1-m)(1-a\alpha(1-m))+\alpha c(1-m)+\alpha\beta rk(1-m),$$

$$\hat{C}=(c+\beta rk)(1-a\alpha(1-m))+r(1-m)(br(1-m)-a\beta),$$

$$\hat{D}=brk(1-m)-a(c+\beta rk).$$

Obviously, $\hat{A}>0$. If the parameters satisfy

$$\frac{a\beta}{b}<1-m<\min\{\frac{1}{a\alpha}, \frac{a}{b}(\beta+\frac{c}{rk})\}, \tag{H}$$

namely

$$a\alpha(1-m)<1,\ b(1-m)>a\beta,\ brk(1-m)<a(c+\beta rk),$$

then $\hat{B}>0$, $\hat{C}>0$, $\hat{D}<0$. According to Lemma 1, it follows that the equation (7) has a unique positive solution $\tilde{u}$, and thus there exists a unique positive constant equilibrium $(\tilde{u}, \tilde{v})$, where $\tilde{v}=r((1-m)\tilde{u}+k)/c$.

Lemma 2 If (H) holds, then the system (3) has a unique positive constant equilibrium $\tilde{\boldsymbol{w}}=(\tilde{u}, \tilde{v})$, which satisfies the equations (6).

The stability of the trivial solution $(0, 0)$ and the semi-trivial solutions $(a, 0)$ and $(0, rk/c)$ of system (3) is discussed below.

Theorem 1 (i) The trivial solution $(0, 0)$ is unconditionally unstable;

(ii) The semi-trivial solution $(a, 0)$ is unconditionally unstable;

(iii) If $1-m<\dfrac{a}{b}(\beta+\dfrac{c}{rk})$, then the semi-trivial solution $(0, \dfrac{rk}{c})$ is unstable. If $1-m>\dfrac{a}{b}(\beta+\dfrac{c}{rk})$, then the semi-trivial solution $(0, rk/c)$ is stable.

Proof We only prove part (iii) in detail, since the proofs of parts (i) and (ii) are similar.

The Jacobian matrix of system (3) at the semi-trivial solution $(0, rk/c)$ is

$$\boldsymbol{J}_0=\begin{pmatrix} a-\dfrac{brk(1-m)}{c+\beta rk} & 0 \\ \dfrac{r^2(1-m)}{c} & -r \end{pmatrix}.$$

The characteristic polynomial of $\boldsymbol{J}_0$ is

$$\varphi_0(\lambda)=\lambda^2-(a-\frac{brk(1-m)}{c+\beta rk}-r)\lambda-r(a-\frac{brk(1-m)}{c+\beta rk}).$$

If $1-m<(a/b)(\beta+(c/rk))$, then $\varphi_0(\lambda)$ has a positive root and a negative root. Hence the semi-trivial solution $(0, rk/c)$ is unstable. If $1-m>(a/b)(\beta+(c/rk))$, then $\varphi_0(\lambda)$ has two negative roots. Hence the semi-trivial solution $(0, rk/c)$ is stable.

Theorem 2 If (H) holds, then the unique positive constant equilibrium $\tilde{\boldsymbol{w}}=(\tilde{u}, \tilde{v})$ of system (3) is locally asymptotically stable.

Proof By Lemma 2, system (3) admits a unique positive constant equilibrium $\tilde{\boldsymbol{w}}=(\tilde{u}, \tilde{v})$. We next examine its local asymptotic stability.

The Jacobian matrix of system (3) at $\tilde{\boldsymbol{w}}=(\tilde{u}, \tilde{v})$ is

$$\boldsymbol{J}=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where

$$\begin{cases} a_{11}=a-2\tilde{u}-\dfrac{b\tilde{v}(1-m)}{(1+\alpha(1-m)\tilde{u})^2(1+\beta\tilde{v})} \\ \quad\quad =\dfrac{\tilde{u}[a\alpha(1-m)-1-2\alpha(1-m)\tilde{u}]}{1+\alpha(1-m)\tilde{u}}, \\ a_{12}=-\dfrac{b(1-m)\tilde{u}}{(1+\alpha(1-m)\tilde{u})(1+\beta\tilde{v})^2}, \\ a_{21}=\dfrac{c\tilde{v}^2(1-m)}{((1-m)\tilde{u}+k)^2}=\dfrac{r^2(1-m)}{c}, \\ a_{22}=r-\dfrac{2c\tilde{v}}{(1-m)\tilde{u}+k}=-r. \end{cases} \tag{8}$$

It is easy to see $a_{12}<0$, $a_{21}>0$, $a_{22}<0$. It follows

that $a_{11}<0$ from $1-m<\dfrac{1}{a\alpha}$.

The characteristic polynomial of $\boldsymbol{J}$ is

$$\varphi_1(\lambda)=\lambda^2-\lambda\mathrm{tr}(\boldsymbol{J})+\det(\boldsymbol{J}),$$

where

$$\mathrm{tr}(\boldsymbol{J})=a_{11}+a_{22}<0,\ \det(\boldsymbol{J})=a_{11}a_{22}-a_{12}a_{21}>0.$$

Thus $\tilde{\boldsymbol{w}}$ is locally asymptotically stable.

The main focus of the following study is on the stability of $\tilde{\boldsymbol{w}}=(\tilde{u},\tilde{v})$ in the reaction-diffusion system (4) and, further, in the cross-diffusion system (5).

Let $0=\mu_0<\mu_1<\mu_2<L$ be the eigenvalues of $-\Delta$ under Neumann boundary conditions, and let $E(\mu_i)$ be the corresponding eigenspace. Choose an orthonormal basis $\{\phi_{ij}\}_{j=1}^{\dim E(\mu_i)}$ of $E(\mu_i)$ and define

$$\boldsymbol{X}_{ij}:=\{\boldsymbol{c}\phi_{ij}\,|\,\boldsymbol{c}\in\mathbb{R}^2\},\ \boldsymbol{X}_i:=\oplus_{j=1}^{\dim E(\mu_i)}\boldsymbol{X}_{ij},\ \boldsymbol{X}:=\oplus_{i=0}^{\infty}\boldsymbol{X}_i, \tag{9}$$

where $\boldsymbol{X}=\{\boldsymbol{w}\in[C^1(\bar{\Omega})]^2\,|\,\dfrac{\partial\boldsymbol{w}}{\partial n}=0 \text{ on } \partial\Omega\}$.

Theorem 3 If (H) holds, then the equilibrium $\tilde{\boldsymbol{w}}$ of the diffusive system (4) is locally asymptotically stable.

Proof System (4) is linearized at $\tilde{\boldsymbol{w}}=(\tilde{u},\tilde{v})$ as follows:

$$\boldsymbol{w}_t=(\boldsymbol{D}_1\Delta+\boldsymbol{J})\boldsymbol{w},\ \boldsymbol{D}_1=\begin{pmatrix}d_1&0\\0&d_2\end{pmatrix}.$$

For each $i\geqslant 0$, $\boldsymbol{X}_{ij}$ is invariant under the operator $\boldsymbol{D}_1\Delta+\boldsymbol{J}$; λ is an eigenvalue of $\boldsymbol{D}_1\Delta+\boldsymbol{J}$ if and only if λ is an eigenvalue of $-\mu_i\boldsymbol{D}_1+\boldsymbol{J}$. By calculation, the characteristic polynomial of $-\mu_i\boldsymbol{D}_1+\boldsymbol{J}$ is

$$\psi_i(\lambda)=\lambda^2-\lambda\mathrm{tr}\{-\mu_i\boldsymbol{D}_1+\boldsymbol{J}\}+\det\{-\mu_i\boldsymbol{D}_1+\boldsymbol{J}\},$$

where

$$\mathrm{tr}\{-\mu_i\boldsymbol{D}_1+\boldsymbol{J}\}=-\mu_i(d_1+d_2)+a_{11}+a_{22},$$
$$\det\{-\mu_i\boldsymbol{D}_1+\boldsymbol{J}\}=d_1d_2\mu_i^2-\mu_i(d_1a_{22}+d_2a_{11})+\det(\boldsymbol{J}).$$

Similar to the proof of Theorem 2, we can obtain $\mathrm{tr}\{-\mu_i\boldsymbol{D}_1+\boldsymbol{J}\}<0$, $\det\{-\mu_i\boldsymbol{D}_1+\boldsymbol{J}\}>0$. Hence the equilibrium $\tilde{\boldsymbol{w}}$ of the diffusive system (4) is locally asymptotically stable.

Theorem 4 Suppose (H) holds and $T>0$, where T is defined by (11). If the cross-diffusion coefficient d_{21} exceeds the critical value

$$d_{21}>d_{21}^{+}=\frac{\sigma+2\sqrt{AC}}{\tau},$$

with $\tau=a_{11}\tilde{u}-a_{12}\tilde{v}$ and $\sigma=d_{12}(a_{21}\tilde{u}+r\tilde{v})-a_{11}d_2+rd_1$, then the equilibrium $\tilde{\boldsymbol{w}}$ of the cross-diffusion system (5) is unstable. Here a_{11},a_{12},a_{21} are given by (8), and A,C by (10).

Proof Let $\boldsymbol{w}=(u,v)^{\mathrm{T}}$ and write system (5) as

$$\boldsymbol{w}_t=\nabla\cdot\left(\boldsymbol{\Phi}(\boldsymbol{w})\nabla\boldsymbol{w}\right)+\boldsymbol{F}(\boldsymbol{w}),$$

where $\boldsymbol{\Phi}(\boldsymbol{w})=\begin{pmatrix}d_1+d_{12}v&d_{12}u\\d_{21}v&d_2+d_{21}u\end{pmatrix}$ and $\boldsymbol{F}(\boldsymbol{w})$ represents the reaction terms. Linearizing around the equilibrium $\tilde{\boldsymbol{w}}=(\tilde{u},\tilde{v})$ gives

$$\boldsymbol{w}_t=(\boldsymbol{\Phi}_{\boldsymbol{w}}(\tilde{\boldsymbol{w}})\Delta+\boldsymbol{J})\boldsymbol{w},$$

where

$$\boldsymbol{\Phi}_{\boldsymbol{w}}(\tilde{\boldsymbol{w}})=\begin{pmatrix}d_1+d_{12}\tilde{v}&d_{12}\tilde{u}\\d_{21}\tilde{v}&d_2+d_{21}\tilde{u}\end{pmatrix},\ \boldsymbol{J}=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}.$$

Seeking solutions of the form $\mathrm{e}^{\lambda t+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}$ (where $\boldsymbol{k}\cdot\boldsymbol{k}=\hat{k}^2$) yields the characteristic equation

$$\det(\lambda\boldsymbol{I}+\hat{k}^2\boldsymbol{\Phi}_{\boldsymbol{w}}(\tilde{\boldsymbol{w}})-\boldsymbol{J})=0.$$

Computing the determinant gives the dispersion relation

$$\lambda^2-p(\hat{k}^2)\lambda+q(\hat{k}^2)=0,$$

with

$$p(\hat{k}^2)=\mathrm{tr}(\boldsymbol{J})-\hat{k}^2(d_1+d_2+d_{12}\tilde{v}+d_{21}\tilde{u}),$$
$$q(\hat{k}^2)=A\hat{k}^4-B\hat{k}^2+C,$$

where

$$\begin{cases}A=d_1d_2+d_1d_{21}\tilde{u}+d_2d_{12}\tilde{v},\\B=(a_{11}d_{21}-a_{21}d_{12})\tilde{u}+(a_{22}d_{12}-a_{12}d_{21})\tilde{v}+a_{11}d_2+a_{22}d_1,\\C=\det(\boldsymbol{J})=a_{11}a_{22}-a_{12}a_{21}.\end{cases} \tag{10}$$

From the parameter conditions in Theorem 4, one can verify that $\mathrm{tr}(\boldsymbol{J})<0$ and $C>0$, and clearly $A>0$.

Using $a_{22}=-r$ from (8) and $\tilde{v}=\dfrac{r((1-m)\tilde{u}+k)}{c}$ from the second equation of (6), the coefficient B can be rearranged as

$$B=\tau d_{21}-\sigma,$$

where

$$\tau=a_{11}\tilde{u}-a_{12}\tilde{v},\ \ \sigma=d_{12}(a_{21}\tilde{u}+r\tilde{v})-a_{11}d_2+rd_1.$$

Applying the first equation of system (6), we obtain

$$\tau=\tilde{u}\cdot T,$$

where

$$T=\frac{\alpha(1-m)\tilde{u}(a-\tilde{u})}{1+\alpha(1-m)\tilde{u}}+\frac{a-\tilde{u}}{1+\beta\tilde{v}}-\tilde{u}. \tag{11}$$

Since $\tilde{u}>0$, the condition $\tau>0$ is equivalent to $T>0$. This inequality is assumed to hold in Theorem 4.

Turing instability occurs if there exists $\hat{k}^2>0$ such that $q(\hat{k}^2)<0$. Since $A>0$ and $C>0$, the minimum of $q(\hat{k}^2)$ is attained at $\hat{k}_0^2=B/(2A)$ with value

$$q_{\min}=C-\frac{B^2}{4A}.$$

Thus $q_{\min}<0$ is equivalent to $B>0$ and $B^2>4AC$.

Substituting $B=\tau d_{21}-\sigma$ into $B^2>4AC$ gives

$$(\tau d_{21}-\sigma)^2>4AC.$$

Since $\tau>0$, solving this quadratic inequality yields

$$d_{21} > \frac{\sigma + 2\sqrt{AC}}{\tau} \quad \text{or} \quad d_{21} < \frac{\sigma - 2\sqrt{AC}}{\tau}.$$

The second branch violates $B>0$, hence only the first branch is admissible.

Therefore, define

$$d_{21}^{+} := \frac{\sigma + 2\sqrt{AC}}{\tau}.$$

When $d_{21}>d_{21}^{+}$, we have $B>0$ and $B^2>4AC$, which implies $q_{\min}<0$ and consequently the equilibrium $\tilde{\boldsymbol{w}}$ is unstable.

Remark 1 The condition $d_{21} > d_{21}^{+}$ ensures that $B>0$ and $B^2>4AC$. Consequently, the quadratic $q(\hat{k}^2)=A\hat{k}^4 - B\hat{k}^2 + C$ attains a negative minimum at $\hat{k}_0^2 = B/(2A)$, and there exists a nonempty interval $(\hat{k}_{\min}^2, \hat{k}_{\max}^2) \subset \mathbb{R}^+$ such that $q(\hat{k}^2)<0$ for all $\hat{k}^2 \in (\hat{k}_{\min}^2, \hat{k}_{\max}^2)$.

Since the eigenvalues $\{\mu_i\}_{i\geqslant 0}$ of $-\Delta$ on Ω with homogeneous Neumann boundary condition satisfy $\mu_0 = 0 < \mu_1 \leqslant \mu_2 \leqslant L \to \infty$ and are densely distributed in $\mathbb{R}^+$ as $i \to \infty$, there exist some $i \geqslant 1$ such that $\mu_i \in (\hat{k}_{\min}^2, \hat{k}_{\max}^2)$. For this mode, the dispersion relation yields a positive growth rate, and hence the constant equilibrium $\tilde{\boldsymbol{w}}$ is unstable. Thus, Turing instability occurs.

Remark 2 Theorem 2 and Theorem 3 show that adding diffusion alone does not affect the local asymptotic stability of the positive constant equilibrium $\tilde{\boldsymbol{w}}$. In contrast, Theorem 4 demonstrates that introducing cross-diffusion destabilizes $\tilde{\boldsymbol{w}}$, thereby generating Turing instability.

2 Nonexistence and Existence of Nonconstant Positive Equilibrium

In this section, we investigate the nonexistence of nonconstant positive equilibria for (4) and the existence for (5). The corresponding equilibrium system for (5) is given by:

$$\begin{cases} d_1\Delta u + d_{12}\Delta(uv) + u\left(a - u - \dfrac{b(1-m)v}{(1+\alpha(1-m)u)(1+\beta v)}\right) = 0, & x \in \Omega, \\ d_2\Delta v + d_{21}\Delta(uv) + v\left(r - \dfrac{cv}{(1-m)u+k}\right) = 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{12}$$

For simplicity, denote the parameter set by $\Lambda = \{a, b, c, m, r, k\}$. We proceed in three steps. First, applying the Maximum Principle[3] and Harnack Inequality[28], we derive uniform upper and lower bounds for any nonconstant positive solution. Second, using Young's inequality and Poincaré's inequality, we prove the nonexistence of nonconstant positive solutions for (12) in the absence of cross-diffusion (i. e., $d_{12}=d_{21}=0$). Third, for the cross-diffusion system (12), we use Leray-Schauder degree theory to establish the existence of nonconstant positive solutions.

2.1 A Priori Estimates

This subsection will use the Maximum Principle and Harnack Inequality to give upper and lower bounds for system (12). It starts with the following lemmas.

Lemma 3 (Maximum Principle[3]) Suppose $g(x,w) \in C(\bar{\Omega} \times \mathbb{R}^1)$.

(i) If $w \in C^2(\Omega) \cap C^1(\Omega)$ satisfies $\Delta w(x) + g(x, w(x)) \geqslant 0$, $x \in \Omega$; $\partial_n w \leqslant 0$, $x \in \partial\Omega$ and $w(x_0) = \max_{\bar{\Omega}} w$, then $g(x_0, w(x_0)) \geqslant 0$.

(ii) If $w \in C^2(\Omega) \cap C^1(\Omega)$ satisfies $\Delta w(x) + g(x, w(x)) \leqslant 0$, $x \in \Omega$; $\partial_n w \geqslant 0, x \in \partial\Omega$ and $w(x_0) = \min_{\bar{\Omega}} w$, then $g(x_0, w(x_0)) \leqslant 0$.

Lemma 4 (Harnack Inequality[28]) Suppose that $c(x) \in C(\bar{\Omega})$, and let $w(x) \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be a positive solution to

$$\Delta w(x) + c(x)w(x) = 0, x \in \Omega; \partial_n w = 0, x \in \partial\Omega.$$

Then there exists a positive constant $C^* = C^*(\|c\|_\infty, \Omega)$ such that $\max_{\bar{\Omega}} w \leqslant C^* \min_{\bar{\Omega}} w$.

Theorem 5 Assume that

$$\frac{d_{12}}{d_1} < \frac{c}{ar(1-m)}.$$

Then there exist positive constants $C_1^* = C_1^*(d_1, d_{12}, \Lambda)$ and $C_2^* = C_2^*(d_1, d_{12}, \Lambda)$ such that for any positive solution (u, v) of system (12),

$$\max_{\bar{\Omega}} u \leqslant C_1^*, \qquad \max_{\bar{\Omega}} v \leqslant C_2^*.$$

Proof Let $\varphi = d_1 u + d_{12} uv$, $\psi = d_2 v + d_{21} uv$. Then system (13) can be written as

$$\begin{cases} -\Delta\varphi = \dfrac{a - u - \dfrac{b(1-m)v}{(1+\alpha(1-m)u)(1+\beta v)}}{d_1 + d_{12}v}\varphi, & x \in \Omega, \\ -\Delta\psi = \dfrac{r - \dfrac{cv}{(1-m)u+k}}{d_2 + d_{21}u}\psi, & x \in \Omega, \\ \dfrac{\partial\varphi}{\partial n} = \dfrac{\partial\psi}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{13}$$

Take $x_1 \in \bar{\Omega}$ such that $\psi(x_1) = \max_{\bar{\Omega}} \psi$. From the second equation of (13) and Lemma 3 (i) we obtain

$$r - \frac{cv(x_1)}{(1-m)u(x_1)+k} \geqslant 0,$$

which implies

$$v(x_1)\leqslant \frac{r((1-m)u(x_1)+k)}{c}.$$

Since $u(x_1)\leqslant \max\limits_{\bar{\Omega}} u$, it follows that

$$\max_{\bar{\Omega}} v\leqslant \frac{r}{c}((1-m)\max_{\bar{\Omega}} u+k). \tag{14}$$

Let $x_2\in\bar{\Omega}$ satisfy $\varphi(x_2)=\max_{\bar{\Omega}}\varphi$. By applying Lemma 3 (i) to the first equation of (13), we have

$$a-u(x_2)-\frac{b(1-m)v(x_2)}{(1+\alpha(1-m)u(x_2))(1+\beta v(x_2))}\geqslant 0,$$

Hence $u(x_2)\leqslant a$. Using $\varphi=(d_1+d_{12}v)u$ we obtain

$$\begin{aligned}\max_{\bar{\Omega}} u&\leqslant \frac{1}{d_1}\max_{\bar{\Omega}}\varphi=u(x_2)+\frac{d_{12}}{d_1}u(x_2)v(x_2)\\&\leqslant a+\frac{d_{12}}{d_1}a\max_{\bar{\Omega}} v.\end{aligned}$$

Denote $U=\max_{\bar{\Omega}} u$ and $V=\max_{\bar{\Omega}} v$; the above inequality becomes

$$U\leqslant a+\frac{d_{12}}{d_1}aV. \tag{15}$$

Substituting (14) into (15) yields

$$U\leqslant a+\frac{d_{12}}{d_1}a\frac{r}{c}((1-m)U+k).$$

Set $A=\dfrac{d_{12}}{d_1}\dfrac{ar}{c}$. Then

$$U\leqslant a+Ak+A(1-m)U,$$

or equivalently,

$$(1-A(1-m))U\leqslant a+Ak.$$

The hypothesis $\dfrac{d_{12}}{d_1}<\dfrac{c}{ar(1-m)}$ implies $1-A(1-m)>0$. Consequently,

$$U\leqslant \frac{a+Ak}{1-A(1-m)}\triangleq C_1^*.$$

Finally, from (14) we obtain

$$V\leqslant \frac{r}{c}((1-m)C_1^*+k)\triangleq C_2^*.$$

Both C_1^* and C_2^* depend only on d_1, d_{12} and Λ. This completes the proof.

Theorem 6 (Lower bound) Let $\delta>0$ and $D>\dfrac{c}{ar(1-m)}$ be fixed constants. Assume that

$$1-m<\frac{a}{b}(\beta+\frac{c}{rk}).$$

If $d_1, d_2\geqslant\delta$ and $\dfrac{d_{12}}{d_1}, \dfrac{d_{21}}{d_2}\leqslant D$, then there exist positive constants $C_1=C_1(\delta, D, \Lambda)$ and $C_2=C_2(\delta, D, \Lambda)$ such that for any positive solution (u, v) of system (12),

$$\min_{\bar{\Omega}} u\geqslant C_1,\quad \min_{\bar{\Omega}} v\geqslant C_2.$$

Proof From the equations in (13), we have $-\Delta\phi=c_1(x)\phi$ and $-\Delta\psi=c_2(x)\psi$, where

$$c_1(x)=\frac{a-u(x)-\dfrac{b(1-m)v(x)}{(1+\alpha(1-m)u(x))(1+\beta v(x))}}{d_1+d_{12}v(x)},$$

$$c_2(x)=\frac{r-\dfrac{cv(x)}{(1-m)u(x)+k}}{d_2+d_{21}u(x)}.$$

Since $0<u\leqslant C_1^*$ and $0<v\leqslant C_2^*$, we have $d_1+d_{12}v\geqslant d_1>0$ and $d_2+d_{21}u\geqslant d_2>0$, and both numerators are bounded. Therefore $\|c_1\|_\infty\leqslant M_1$ and $\|c_2\|_\infty\leqslant M_2$ for some constants $M_1=M_1(C_1^*, C_2^*, \Lambda)$ and $M_2=M_2(C_1^*, C_2^*, \Lambda)$. By Lemma 4, there exist positive constants $\widehat{C_1}=\widehat{C_1}(M_1, \delta, \Lambda)$ and $\widehat{C_2}=\widehat{C_2}(M_2, \delta, \Lambda)$ such that

$$\max_{\bar{\Omega}}\varphi\leqslant \widehat{C_1}\min_{\bar{\Omega}}\varphi,\quad \max_{\bar{\Omega}}\psi\leqslant \widehat{C_2}\min_{\bar{\Omega}}\psi.$$

Consequently, we have the following estimates:

$$\begin{aligned}\frac{\max_{\bar{\Omega}} u(x)}{\min_{\bar{\Omega}} u(x)}&\leqslant \frac{\max_{\bar{\Omega}}\varphi}{\min_{\bar{\Omega}}\varphi}\frac{\max_{\bar{\Omega}}(d_1+d_{12}v)}{\min_{\bar{\Omega}}(d_1+d_{12}v)}\\&\leqslant \widehat{C_1}(1+\frac{d_{12}}{d_1}C_2^*)\leqslant \widehat{C_1}(1+DC_2^*)\triangleq \widetilde{C_1},\end{aligned} \tag{16}$$

$$\begin{aligned}\frac{\max_{\bar{\Omega}} v(x)}{\min_{\bar{\Omega}} v(x)}&\leqslant \frac{\max_{\bar{\Omega}}\psi}{\min_{\bar{\Omega}}\psi}\frac{\max_{\bar{\Omega}}(d_{21}u+d_2)}{\min_{\bar{\Omega}}(d_{21}u+d_2)}\\&\leqslant \widehat{C_2}(1+\frac{d_{21}}{d_2}C_1^*)\leqslant \widehat{C_2}(1+DC_1^*)\triangleq \widetilde{C_2}.\end{aligned} \tag{17}$$

By contradiction, assuming that the conclusion does not hold, there exists a sequence $\{d_{1,n}, d_{2,n}, d_{12,n}, d_{21,n}\}_{n=1}^{\infty}$, $d_{1,n}, d_{2,n}\geqslant\delta$, $d_{12,n}, d_{21,n}\in(0,\infty)$ such that the positive solution (u_n, v_n) of system (12) satisfies

$$\begin{cases}-\Delta(d_{1,n}u_n+d_{12,n}u_nv_n)\\\quad=u_n\left(a-u_n-\dfrac{b(1-m)v_n}{(1+\alpha(1-m)u_n)(1+\beta v_n)}\right), & x\in\Omega,\\-\Delta(d_{2,n}v_n+d_{21,n}u_nv_n)=v_n\left(r-\dfrac{cv_n}{(1-m)u_n+k}\right), & x\in\Omega,\\\dfrac{\partial u_n}{\partial n}=\dfrac{\partial v_n}{\partial n}=0, & x\in\partial\Omega,\end{cases} \tag{18}$$

and $\min\limits_{\bar{\Omega}} u_n\to 0$ or $\min\limits_{\bar{\Omega}} v_n\to 0 (n\to\infty)$. By standard elliptic regularity theory, there exists a subsequence of $\{(u_n, v_n)\}_{n=1}^{\infty}$, still denoted by $\{(u_n, v_n)\}_{n=1}^{\infty}$, and nonnegative functions $u, v\in C^2(\bar{\Omega})$ such that $(u_n, v_n)\to(u, v)$ as $n\to\infty$. Assume further that

$$(d_{1,n}, d_{2,n}, d_{12,n}, d_{21,n})\to(d_1, d_2, d_{12}, d_{21}),$$

where $(d_1, d_2, d_{12}, d_{21})\in[\delta,\infty)\times[\delta,\infty)\times(0,\infty)\times(0,\infty)$. From inequalities (16) and (17) together with the assumption $\min\limits_{\bar{\Omega}} u_n\to 0$ or $\min\limits_{\bar{\Omega}} v_n\to 0$, it follows that as $n\to\infty$,

$$\max_{\bar{\Omega}} u_n\to 0 \text{ or } \max_{\bar{\Omega}} v_n\to 0.$$

We consider the following three cases.

(i) If $\max\limits_{\bar{\Omega}} u_n\to 0$ and $\max\limits_{\bar{\Omega}} v_n\to 0$, then $(u_n, v_n)\to(0, 0)$

uniformly. However, integrating the first equation of (18) over Ω, we obtain

$$0=-\int_{\partial\Omega}\frac{\partial}{\partial n}(d_{1,n}u_n+d_{12,n}u_nv_n)\mathrm{d}S$$
$$=\int_{\Omega}u_n\left(a-u_n-\frac{b(1-m)v_n}{(1+\alpha(1-m)u_n)(1+\beta v_n)}\right)\mathrm{d}x.$$

Since $u_n>0$ and the integrand converges to $a>0$ uniformly as $n\to\infty$, the right-hand side is positive for sufficiently large n, a contradiction. Hence Case (i) cannot occur.

(ii) If $\max_{\bar{\Omega}}u_n\to 0$, $\max_{\bar{\Omega}}v_n\to v^*$, where v^* is a positive constant. Set $\epsilon_n=d_{2,n}v_n+d_{21,n}u_nv_n$. Then ϵ_n satisfies

$$\begin{cases}-\Delta\epsilon_n=\dfrac{r-\dfrac{cv_n}{(1-m)u_n+k}}{d_{2,n}+d_{21,n}u_n}\epsilon_n, & x\in\Omega,\\ \dfrac{\partial\epsilon_n}{\partial n}=0, & x\in\partial\Omega.\end{cases}$$

By the upper bound of u_n, v_n, since $\|\epsilon_n\|_{\infty}\leqslant C$ for any $n\geqslant 1$, there exists a subsequence of ϵ_n which is still denoted by ϵ_n, such that $\epsilon_n\to\epsilon$ in $C^2(\Omega)$. By the definition of ϵ_n and $u_n\to 0$, one can obtain

$$v_n=\frac{\epsilon_n}{d_{2,n}+d_{21,n}u_n}\to\frac{\epsilon}{d_2}\qquad(n\to\infty).$$

Then ϵ satisfies

$$\begin{cases}-d_2^2\Delta\epsilon=\left(d_2r-\dfrac{c\epsilon}{k}\right)\epsilon, & x\in\Omega,\\ \dfrac{\partial\epsilon}{\partial n}=0, & x\in\partial\Omega.\end{cases}$$

If $\epsilon=0$, then $v^*=\dfrac{\epsilon}{d_2}=0$ contradicts $v^*>0$. So $\epsilon=\dfrac{d_2rk}{c}$, $v^*=\dfrac{rk}{c}$, and hence $(u_n,v_n)\to(0,\dfrac{rk}{c})$ uniformly on $\bar{\Omega}$. Since the parameter condition guarantees

$$1-m<\frac{a}{b}(\beta+\frac{c}{rk}),$$

for sufficiently large n and all $x\in\Omega$,

$$a-u_n(x)-\frac{b(1-m)v_n(x)}{(1+\alpha(1-m)u_n(x))(1+\beta v_n(x))}>0.$$

Integrating the first equation of (18) over Ω, we obtain

$$0=-\int_{\partial\Omega}\frac{\partial}{\partial n}(d_{1,n}u_n+d_{12,n}u_nv_n)\mathrm{d}S$$
$$=\int_{\Omega}u_n\left(a-u_n-\frac{b(1-m)v_n}{(1+\alpha(1-m)u_n)(1+\beta v_n)}\right)\mathrm{d}x>0,$$

a contradiction. Therefore Case (ii) cannot occur.

(iii) If $\max\limits_{\bar{\Omega}}u_n\to u^*$ and $\max\limits_{\bar{\Omega}}v_n\to 0$, where u^* is a positive constant. Set $\xi_n=d_{1,n}u_n+d_{12,n}u_nv_n$. Then ξ_n satisfies

$$\begin{cases}-\Delta\xi_n=\dfrac{a-u_n-\dfrac{b(1-m)v_n}{(1+\alpha(1-m)u_n)(1+\beta v_n)}}{d_{1,n}+d_{12,n}v_n}\xi_n, & x\in\Omega,\\ \dfrac{\partial\xi_n}{\partial n}=0, & x\in\partial\Omega.\end{cases}$$

By the upper bounds of u_n and v_n, we have $\|\xi_n\|_{\infty}\leqslant C$ for all $n\geqslant 1$. Thus there exists a subsequence, still denoted by ξ_n, such that $\xi_n\to\xi$ in $C^2(\Omega)$ as $n\to\infty$. From the definition of ξ_n and $v_n\to 0$, we obtain

$$u_n=\frac{\xi_n}{d_{1,n}+d_{12,n}v_n}\to\frac{\xi}{d_1}\ (n\to\infty).$$

Then ξ satisfies the limiting problem

$$\begin{cases}-d_1^2\Delta\xi=(d_1a-\xi)\xi, & x\in\Omega,\\ \dfrac{\partial\xi}{\partial n}=0, & x\in\partial\Omega.\end{cases}$$

If $\xi\equiv 0$, then $u^*=\xi/d_1=0$, contradicting $u^*>0$. Hence $\xi\equiv d_1a$, which gives $u^*=a$, and consequently $(u_n,v_n)\to(a,0)$ uniformly on $\bar{\Omega}$.

Now consider the second equation of (18). Since $v_n\to 0$ and $u_n\to a$, we have

$$r-\frac{cv_n}{(1-m)u_n+k}\to r>0.$$

Thus for sufficiently large n and all $x\in\Omega$,

$$r-\frac{cv_n(x)}{(1-m)u_n(x)+k}>0.$$

Integrating the second equation of (18) over Ω, we obtain

$$0=-\int_{\partial\Omega}\frac{\partial}{\partial n}(d_{2,n}v_n+d_{21,n}u_nv_n)\mathrm{d}S$$
$$=\int_{\Omega}v_n\left(r-\frac{cv_n}{(1-m)u_n+k}\right)\mathrm{d}x>0,$$

a contradiction. Therefore Case (iii) cannot occur.

2.2 Nonexistence of Nonconstant Positive Equilibrium

In this subsection, we show that system (12) has no nonconstant positive solutions in the case of diffusion only (not cross-diffusion). The proof uses Young's inequality and Poincaré's inequality.

Theorem 7 If $d_{12}=d_{21}=0$, then there exist positive constants $d_1^*=d_1^*(\delta,D,\Lambda)$ and $d_2^*=d_2^*(\delta,D,\Lambda)$ such that for any $d_1\geqslant d_1^*$ and $d_2\geqslant d_2^*$, system (12) has no nonconstant positive solution.

Proof Assume that (u,v) is a positive solution of (12) with $d_{12}=d_{21}=0$. By Theorems 5 and 6, there exist constants $c_1,c_2,C_1,C_2>0$, depending only on δ,D,Λ, such that

$$c_1\leqslant u\leqslant C_1,\ c_2\leqslant v\leqslant C_2\ \text{in}\ \bar{\Omega}.$$

Denote the average values of u and v over Ω by

$$\bar{u}=\frac{1}{|\Omega|}\int_{\Omega}u\mathrm{d}x,\quad\bar{v}=\frac{1}{|\Omega|}\int_{\Omega}v\mathrm{d}x.$$

Multiplying the first equation of (12) by $u-\bar{u}$ and integrating over Ω, we obtain

$$d_1\int_{\Omega}|\nabla u|^2\mathrm{d}x=\int_{\Omega}(u-\bar{u})(f(u,v)-f(\bar{u},\bar{v}))\mathrm{d}x,$$

where $f(u,v)=u(a-u-\frac{b(1-m)v}{(1+\alpha(1-m)u)(1+\beta v)})$. Similarly, from the second equation we have

$$d_2\int_\Omega|\nabla v|^2\mathrm{d}x=\int_\Omega(v-\bar v)(g(u,v)-g(\bar u,\bar v))\mathrm{d}x,$$

with $g(u,v)=v(r-\frac{cv}{(1-m)u+k})$.

Since f and g are C^1 on the closed rectangle $[c_1,C_1]\times[c_2,C_2]$, there exist constants $L_1,L_2,L_3,L_4>0$ such that

$$|f(u,v)-f(\bar u,\bar v)|\leqslant L_1|u-\bar u|+L_2|v-\bar v|,$$
$$|g(u,v)-g(\bar u,\bar v)|\leqslant L_3|u-\bar u|+L_4|v-\bar v|.$$

Fix an arbitrary $\varepsilon>0$. By Young's inequality,

$$|u-\bar u||v-\bar v|\leqslant\frac{\varepsilon}{2}(u-\bar u)^2+\frac{1}{2\varepsilon}(v-\bar v)^2.$$

Hence,

$$\begin{aligned}&\int_\Omega(u-\bar u)(f(u,v)-f(\bar u,\bar v))\mathrm{d}x\\&\leqslant\int_\Omega\left(L_1(u-\bar u)^2+L_2|u-\bar u||v-\bar v|\right)\mathrm{d}x\\&\leqslant\int_\Omega\left[\left(L_1+\frac{L_2\varepsilon}{2}\right)(u-\bar u)^2+\frac{L_2}{2\varepsilon}(v-\bar v)^2\right]\mathrm{d}x,\\&\int_\Omega(v-\bar v)(g(u,v)-g(\bar u,\bar v))\mathrm{d}x\\&\leqslant\int_\Omega(L_3|u-\bar u||v-\bar v|+L_4(v-\bar v)^2)\mathrm{d}x\\&\leqslant\int_\Omega\left[\frac{L_3\varepsilon}{2}(u-\bar u)^2+\left(L_4+\frac{L_3}{2\varepsilon}\right)(v-\bar v)^2\right]\mathrm{d}x.\end{aligned}$$

Adding the two inequalities, we obtain

$$d_1\int_\Omega|\nabla u|^2\mathrm{d}x+d_2\int_\Omega|\nabla v|^2\mathrm{d}x\leqslant\int_\Omega[A(u-\bar u)^2+B(v-\bar v)^2]\mathrm{d}x,$$

where

$$A=L_1+\frac{\varepsilon}{2}(L_2+L_3),\quad B=L_4+\frac{1}{2\varepsilon}(L_2+L_3).$$

Applying the Poincaré's inequality $\mu_1\int_\Omega(w-\bar w)^2\mathrm{d}x\leqslant\int_\Omega|\nabla w|^2\mathrm{d}x$, we have

$$\begin{aligned}&d_1\mu_1\int_\Omega(u-\bar u)^2\mathrm{d}x+d_2\mu_1\int_\Omega(v-\bar v)^2\mathrm{d}x\\&\leqslant A\int_\Omega(u-\bar u)^2\mathrm{d}x+B\int_\Omega(v-\bar v)^2\mathrm{d}x.\end{aligned}$$

Now define

$$d_1^*=\frac{A}{\mu_1}+1,\quad d_2^*=\frac{B}{\mu_1}+1.$$

If $d_1\geqslant d_1^*$ and $d_2\geqslant d_2^*$, then $A<d_1\mu_1$ and $B<d_2\mu_1$. Consequently,

$$(d_1\mu_1-A)\int_\Omega(u-\bar u)^2\mathrm{d}x+(d_2\mu_1-B)\int_\Omega(v-\bar v)^2\mathrm{d}x\leqslant0,$$

which forces

$$\int_\Omega(u-\bar u)^2\mathrm{d}x=\int_\Omega(v-\bar v)^2\mathrm{d}x=0.$$

Hence $u\equiv\bar u$ and $v\equiv\bar v$ are constants. Therefore, no nonconstant positive solution exists.

2.3 Existence of Nonconstant Positive Equilibrium

This subsection uses Leray-Schauder degree theory to establish the existence of nonconstant positive solutions to the system (12) when the cross-diffusion coefficients d_{12} and d_{21} are introduced while keeping d_1 and d_2 fixed.

For ease of discussion, rewrite the system (12) as

$$\begin{cases}-\Delta\boldsymbol{\Phi}(\boldsymbol{w})=\boldsymbol{Z}(\boldsymbol{w}), & x\in\Omega,\\ \dfrac{\partial\boldsymbol{w}}{\partial n}=0, & x\in\partial\Omega,\end{cases}\tag{19}$$

where $\boldsymbol{w}=(u,v)^{\mathrm T}$, $\boldsymbol{\Phi}(\boldsymbol{w})=(\varphi,\psi)^{\mathrm T}$, $\varphi=d_1u+d_{12}uv$, $\psi=d_2v+d_{21}uv$, $\boldsymbol{Z}(\boldsymbol{w})=(f(u,v),g(u,v))^{\mathrm T}$. Define

$$\boldsymbol{X}=\left\{\boldsymbol{w}\in[C^1(\bar\Omega)]^2\mid\frac{\partial\boldsymbol{w}}{\partial n}=0\text{ on }\partial\Omega\right\},$$
$$\boldsymbol{X}^+=\left\{\boldsymbol{w}\in\boldsymbol{X}\mid u(x)>0,\ v(x)>0\text{ for all }x\in\bar\Omega\right\},$$
$$H(\rho)=\left\{\boldsymbol{w}\in\boldsymbol{X}\mid\rho^{-1}<u(x),v(x)<\rho\text{ for all }x\in\bar\Omega\right\}.$$

By Theorems 5 and 6 there exists a positive constant ρ (depending only on δ,D,Λ) such that all positive solutions of (12) lie in $H(\rho)$.

Since

$$\boldsymbol{\Phi}_{\boldsymbol{w}}(\tilde{\boldsymbol{w}})=\begin{pmatrix}d_1+d_{12}\tilde v & d_{12}\tilde u\\ d_{21}\tilde v & d_2+d_{21}\tilde u\end{pmatrix}.\tag{20}$$

For any positive solution (u,v), we have $\det\boldsymbol{\Phi}_{\boldsymbol{w}}(\boldsymbol{w})>0$, and consequently $\det\boldsymbol{\Phi}_{\boldsymbol{w}}^{-1}(\boldsymbol{w})>0$. Therefore, $\boldsymbol{w}$ is a positive solution of (19) if and only if

$$\begin{aligned}\boldsymbol{G}(\boldsymbol{w}):=\boldsymbol{w}-(\boldsymbol{I}-\Delta)^{-1}\{&\boldsymbol{\Phi}_{\boldsymbol{w}}^{-1}(\boldsymbol{w})[\boldsymbol{Z}(\boldsymbol{w})+\nabla\boldsymbol{w}\boldsymbol{\Phi}_{\boldsymbol{ww}}(\boldsymbol{w})\cdot\nabla\boldsymbol{w}]\\&+\boldsymbol{w}\}=\boldsymbol{0},\end{aligned}\tag{21}$$

with $\boldsymbol{w}\in\boldsymbol{X}^+$. Here $(\boldsymbol{I}-\Delta)^{-1}$ denotes the inverse operator of $\boldsymbol{I}-\Delta$ under homogeneous Neumann boundary conditions, which is a compact resolvent operator. The term $\nabla\boldsymbol{w}\boldsymbol{\Phi}_{\boldsymbol{ww}}(\boldsymbol{w})\cdot\nabla\boldsymbol{w}$ represents the contraction of the bilinear form induced by the second Fréchet derivative $\boldsymbol{\Phi}_{\boldsymbol{ww}}(\boldsymbol{w})$ with the gradient $\nabla\boldsymbol{w}$. Defining

$$\boldsymbol{K}(\boldsymbol{w}):=(\boldsymbol{I}-\Delta)^{-1}\{\boldsymbol{\Phi}_{\boldsymbol{w}}^{-1}(\boldsymbol{w})[\boldsymbol{Z}(\boldsymbol{w})+\nabla\boldsymbol{w}\boldsymbol{\Phi}_{\boldsymbol{ww}}(\boldsymbol{w})\cdot\nabla\boldsymbol{w}]+\boldsymbol{w}\},$$

we have $\boldsymbol{G}=\boldsymbol{I}-\boldsymbol{K}$. Since $(\boldsymbol{I}-\Delta)^{-1}$ is a compact resolvent operator and the remaining terms (i. e., $\boldsymbol{\Phi}_{\boldsymbol{w}}^{-1}(\boldsymbol{w})$, $\boldsymbol{Z}(\boldsymbol{w})$, $\boldsymbol{\Phi}_{\boldsymbol{ww}}(\boldsymbol{w})$ and $\nabla\boldsymbol{w}$) are continuous and bounded on bounded subsets of $\boldsymbol{X}^+$, the operator $\boldsymbol{K}$ is compact. Hence, $\boldsymbol{G}$ is a compact perturbation of the identity operator.

Since $H(\rho)$ is a bounded open neighborhood in the Banach space $\boldsymbol{X}^+$, the Leray-Schauder degree $\deg(\boldsymbol{G},0,H(\rho))$ is well-defined provided that $\boldsymbol{G}(\boldsymbol{w})\neq\boldsymbol{0}$ for

all $\boldsymbol{w}\in\partial H(\rho)$. Moreover, the Leray-Schauder degree is homotopy invariant under admissible homotopies that preserve the nonvanishing property of $\boldsymbol{G}$ on $\partial H(\rho)$. In addition, if $\deg(\boldsymbol{G},0,H(\rho))\neq 0$, then there exists at least one zero of $\boldsymbol{G}$ inside $H(\rho)$, which corresponds to a positive solution of (19).

A straightforward Fréchet differentiation of $\boldsymbol{G}$ at $\tilde{\boldsymbol{w}}$ gives

$$D_{\boldsymbol{w}}\boldsymbol{G}(\tilde{\boldsymbol{w}})=\boldsymbol{I}-(\boldsymbol{I}-\Delta)^{-1}(\boldsymbol{\Phi}_{\boldsymbol{w}}^{-1}(\tilde{\boldsymbol{w}})\boldsymbol{Z}_{\boldsymbol{w}}(\tilde{\boldsymbol{w}})+\boldsymbol{I}),$$

where we have used that $\nabla\tilde{\boldsymbol{w}}=0$ and $\boldsymbol{Z}(\tilde{\boldsymbol{w}})=\boldsymbol{0}$ to eliminate terms containing $\nabla\tilde{\boldsymbol{w}}$ or $\boldsymbol{Z}(\tilde{\boldsymbol{w}})$. If $D_{\boldsymbol{w}}\boldsymbol{G}(\tilde{\boldsymbol{w}})$ is invertible, then the index of $\boldsymbol{G}$ at $\tilde{\boldsymbol{w}}$ can be defined as

$$\operatorname{index}(\boldsymbol{G}(\cdot),\tilde{\boldsymbol{w}})=(-1)^{\zeta},$$

where ζ is the sum of the algebraic multiplicities of all eigenvalues of $D_{\boldsymbol{w}}\boldsymbol{G}(\tilde{\boldsymbol{w}})$ with negative real parts. According to Leray-Schauder degree theory, if $\boldsymbol{G}(\boldsymbol{w})\neq 0$ in ∂H, then $\deg(\boldsymbol{G}(\cdot),0,H)$ is equal to the sum of all isolated point indices of $H(\rho)$ with $\boldsymbol{G}=0$.

By the decomposition of $\boldsymbol{X}$ into $\boldsymbol{X}_i$ (see (9)), for each $i\geqslant 0$, $D_{\boldsymbol{w}}\boldsymbol{G}(\tilde{\boldsymbol{w}})$ acts on $\boldsymbol{X}_i$ as

$$\begin{aligned}D_{\boldsymbol{w}}\boldsymbol{G}(\tilde{\boldsymbol{w}})|_{X_i}&=\boldsymbol{I}-\frac{1}{1+\mu_i}(\boldsymbol{\Phi}_{\boldsymbol{w}}^{-1}(\tilde{\boldsymbol{w}})\boldsymbol{Z}_{\boldsymbol{w}}(\tilde{\boldsymbol{w}})+\boldsymbol{I})\\&=\frac{1}{1+\mu_i}(\mu_i\boldsymbol{I}-\boldsymbol{\Phi}_{\boldsymbol{w}}^{-1}(\tilde{\boldsymbol{w}})\boldsymbol{Z}_{\boldsymbol{w}}(\tilde{\boldsymbol{w}})).\end{aligned}$$

Therefore, $D_{\boldsymbol{w}}\boldsymbol{G}(\tilde{\boldsymbol{w}})$ is invertible if and only if for each $i\geqslant 0$, the matrix $\mu_i\boldsymbol{I}-\boldsymbol{\Phi}_{\boldsymbol{w}}^{-1}(\tilde{\boldsymbol{w}})\boldsymbol{Z}_{\boldsymbol{w}}(\tilde{\boldsymbol{w}})$ is non-singular.

Denote

$$M(\mu):=M(\tilde{\boldsymbol{w}};\mu)\triangleq\det(\mu\boldsymbol{I}-\boldsymbol{\Phi}_{\boldsymbol{w}}^{-1}(\tilde{\boldsymbol{w}})\boldsymbol{Z}_{\boldsymbol{w}}(\tilde{\boldsymbol{w}})).\quad(22)$$

It can be shown that if $M(\mu_i)\neq 0$, then for each $j=1,\cdots,\dim E(\mu_i)$, the number of eigenvalues of $D_{\boldsymbol{w}}\boldsymbol{G}(\tilde{\boldsymbol{w}})$ on $\boldsymbol{X}_{ij}$ with negative real part (counting algebraic multiplicities) is odd if and only if $M(\mu_i)<0$.

Thus we obtain the following result.

Lemma 5 If $M(\mu_i)\neq 0$ for all $i\geqslant 0$, then

$$\operatorname{index}(\boldsymbol{G}(\cdot),\tilde{\boldsymbol{w}})=(-1)^{\gamma},\quad\gamma=\sum_{\substack{i\geqslant 0\\ M(\mu_i)<0}}\dim E(\mu_i).$$

In order to compute $\operatorname{index}(\boldsymbol{G}(\cdot),\tilde{\boldsymbol{w}})$, it is necessary to discuss the sign of $M(\mu_i)$. From the formula (22) we have

$$M(\mu)=\det\{\boldsymbol{\Phi}_{\boldsymbol{w}}^{-1}(\tilde{\boldsymbol{w}})\}\det\{\mu\boldsymbol{\Phi}_{\boldsymbol{w}}(\tilde{\boldsymbol{w}})-\boldsymbol{Z}_{\boldsymbol{w}}(\tilde{\boldsymbol{w}})\}.\quad(23)$$

Since $\det\{\boldsymbol{\Phi}_{\boldsymbol{w}}^{-1}(\tilde{\boldsymbol{w}})\}>0$, it suffices to study the sign of $\det\{\mu\boldsymbol{\Phi}_{\boldsymbol{w}}(\tilde{\boldsymbol{w}})-\boldsymbol{Z}_{\boldsymbol{w}}(\tilde{\boldsymbol{w}})\}$. A direct calculation yields

$$\det\{\mu\boldsymbol{\Phi}_{\boldsymbol{w}}(\tilde{\boldsymbol{w}})-\boldsymbol{Z}_{\boldsymbol{w}}(\tilde{\boldsymbol{w}})\}=A\mu^2-B\mu+C\triangleq:\mathcal{B}(\mu;d_{21}),\quad(24)$$

where A,B,C are given by equations (10). Let $\bar{\mu}_1,\bar{\mu}_2$ be the two roots of $\mathcal{B}(\mu;d_{21})=0$, and $\operatorname{Re}\{\bar{\mu}_1\}\leqslant\operatorname{Re}\{\bar{\mu}_2\}$. If $1-m<1/a\alpha$, then $C>0$ and $\bar{\mu}_1\bar{\mu}_2=C/A>0$, so the two roots have the same sign. Furthermore, there are the following limits

$$\lim_{d_{21}\to\infty}\frac{A}{d_{21}}=d_1\tilde{u}>0,\ \lim_{d_{21}\to\infty}\frac{B}{d_{21}}=a_{11}\tilde{u}-a_{12}\tilde{v}=\tilde{u}T,$$

$$\lim_{d_{21}\to\infty}\frac{\mathcal{B}(\mu;d_{21})}{d_{21}}=\mu\tilde{u}(d_1\mu-T),$$

where T is given by (11). Consequently, we have the following conclusions.

Lemma 6 Suppose (H) holds and $T>0$. Then there exists a positive constant d_{21}^* such that when $d_{21}\geqslant d_{21}^*$, the two roots $\bar{\mu}_1(d_{21}),\bar{\mu}_2(d_{21})$ of $\mathcal{B}(\mu;d_{21})=0$ are real numbers and satisfy

$$\begin{cases}\lim\limits_{d_{21}\to\infty}\bar{\mu}_1(d_{21})=0,\\ \lim\limits_{d_{21}\to\infty}\bar{\mu}_2(d_{21})=\dfrac{T}{d_1}\triangleq\bar{\mu}>0.\end{cases}\quad(25)$$

Moreover

$$\begin{cases}\mathcal{B}(\mu;d_{21})<0, & \mu\in(\bar{\mu}_1(d_{21}),\bar{\mu}_2(d_{21})),\\ \mathcal{B}(\mu;d_{21})>0, & \mu\in(-\infty,\bar{\mu}_1(d_{21}))\cup(\bar{\mu}_2(d_{21}),+\infty).\end{cases}\quad(26)$$

Theorem 8 Assume (H) holds, and $T>0$. Let d_1,d_2,d_{12} be fixed positive constants. If $\bar{\mu}\in(\mu_n,\mu_{n+1})$ for some $n\geqslant 1$ and $\gamma_n=\sum_{i=1}^{n}\dim E(\mu_i)$ is odd, then there exists $d_{21}^*>0$ such that for all $d_{21}\geqslant d_{21}^*$, system (19) has at least one nonconstant positive solution.

Proof By Lemma 6, there exists a positive constant d_{21}^* such that (26) holds when $d_{21}\geqslant d_{21}^*$, and

$$0=\mu_0<\bar{\mu}_1(d_{21})<\bar{\mu}_2(d_{21}),\quad\bar{\mu}_2(d_{21})\in(\mu_n,\mu_{n+1}).\quad(27)$$

Assume, for contradiction, that for some $d_{21}\geqslant d_{21}^*$ system (19) has no nonconstant positive solution.

Fix $d_{21}\geqslant d_{21}^*$. Choose $\hat{d}_1\geqslant d_1^*$ and $\hat{d}_2\geqslant d_2^*$, where d_1^* and d_2^* are given by Theorem 7. Define

$$\boldsymbol{\Phi}(t,\boldsymbol{w})=\begin{pmatrix}[td_1+(1-t)\hat{d}_1]u+td_{12}uv\\ [td_2+(1-t)\hat{d}_2]v+td_{21}uv\end{pmatrix},$$

where $t\in[0,1]$. Consider the following system of equations

$$\begin{cases}-\Delta\boldsymbol{\Phi}(t;\boldsymbol{w})=\boldsymbol{Z}(\boldsymbol{w}), & x\in\Omega,\\ \dfrac{\partial\boldsymbol{w}}{\partial n}=0, & x\in\partial\Omega.\end{cases}\quad(28)$$

Clearly, $\boldsymbol{w}$ is a nonconstant positive solution of the system (19) if and only if it is a positive solution of the system (28) at $t=1$. For any $t\in[0,1]$, $\tilde{\boldsymbol{w}}$ is the unique positive constant solution to (28), and $\boldsymbol{w}$ is a positive solution of the system (28) if and only if

$$\begin{aligned}\boldsymbol{G}(t;\boldsymbol{w})\triangleq\boldsymbol{w}-(\boldsymbol{I}-\Delta)^{-1}\{&\boldsymbol{\Phi}_{\boldsymbol{w}}^{-1}(t;\boldsymbol{w})[\boldsymbol{Z}(\boldsymbol{w})\\&+\nabla\boldsymbol{w}\boldsymbol{\Phi}_{\boldsymbol{www}}(t;\boldsymbol{w})\cdot\nabla\boldsymbol{w}]+\boldsymbol{w}\}=0,\ \boldsymbol{w}\in\boldsymbol{X}^{+}.\end{aligned}$$

Clearly, $\boldsymbol{G}(1;\boldsymbol{w})=\boldsymbol{G}(\boldsymbol{w})$. By Theorem 7 the equation $\boldsymbol{G}(0;\boldsymbol{w})=0$ has only the constant positive solution $\tilde{\boldsymbol{w}}$.

A direct calculation yields the Fréchet derivative of $\boldsymbol{G}(t;\boldsymbol{w})$ at $\tilde{\boldsymbol{w}}$:

$$D_{\boldsymbol{w}}\boldsymbol{G}(t;\tilde{\boldsymbol{w}})=\boldsymbol{I}-(\boldsymbol{I}-\Delta)^{-1}\{\boldsymbol{\Phi}_{\boldsymbol{w}}^{-1}(t;\tilde{\boldsymbol{w}})\boldsymbol{Z}_{\boldsymbol{w}}(\tilde{\boldsymbol{w}})+\boldsymbol{I}\}.$$

Let $\tilde{\boldsymbol{\Phi}}(\boldsymbol{w})=\boldsymbol{\Phi}(0;\boldsymbol{w})$; in particular,

$$D_{\boldsymbol{w}}\boldsymbol{G}(0;\tilde{\boldsymbol{w}})=\boldsymbol{I}-(\boldsymbol{I}-\Delta)^{-1}\{\tilde{\boldsymbol{\Phi}}_{\boldsymbol{w}}^{-1}(\tilde{\boldsymbol{w}})\boldsymbol{Z}_{\boldsymbol{w}}(\tilde{\boldsymbol{w}})+\boldsymbol{I}\},$$

$$D_{\boldsymbol{w}}\boldsymbol{G}(1;\tilde{\boldsymbol{w}})=\boldsymbol{I}-(\boldsymbol{I}-\Delta)^{-1}\{\boldsymbol{\Phi}_{\boldsymbol{w}}^{-1}(\tilde{\boldsymbol{w}})\boldsymbol{Z}_{\boldsymbol{w}}(\tilde{\boldsymbol{w}})+\boldsymbol{I}\}=D_{\boldsymbol{w}}\boldsymbol{G}(\tilde{\boldsymbol{w}}),$$

where $\tilde{\boldsymbol{\Phi}}_{\boldsymbol{w}}(\tilde{\boldsymbol{w}})=\begin{pmatrix}\hat{d}_1 & 0\\ 0 & \hat{d}_2\end{pmatrix}$, $\boldsymbol{\Phi}_{\boldsymbol{w}}(\tilde{\boldsymbol{w}})$ is given by (20) . We set

$$M(t;\mu)=\det\{\boldsymbol{\Phi}_{\boldsymbol{w}}^{-1}(t;\tilde{\boldsymbol{w}})\}\det\{\mu\boldsymbol{\Phi}_{\boldsymbol{w}}(t;\tilde{\boldsymbol{w}})-\boldsymbol{Z}_{\boldsymbol{w}}(\tilde{\boldsymbol{w}})\}.$$

When $t=1$, from (23) and (24) we have

$$M(1;\mu)=M(\mu)=\det\{\boldsymbol{\Phi}_{\boldsymbol{w}}^{-1}(\tilde{\boldsymbol{w}})\}\mathcal{B}(\mu;d_{21}). \tag{29}$$

Since $\det\{\boldsymbol{\Phi}_{\boldsymbol{w}}^{-1}(\tilde{\boldsymbol{w}})\}>0$, the sign of $M(1;\mu_i)$ is the same as that of $\mathcal{B}(\mu_i;d_{21})$. By (26) and (27), we have:

• For $i=0$: $\mu_0=0<\bar{\mu}_1$, hence $\mathcal{B}(0)>0$, and therefore $M(1;\mu_0)>0$;

• For $1\leqslant i\leqslant n$: $\bar{\mu}_1<\mu_i<\bar{\mu}_2$, hence $\mathcal{B}(\mu_i)<0$, and therefore $M(1;\mu_i)<0$;

• For $i\geqslant n+1$: $\mu_i>\bar{\mu}_2$, hence $\mathcal{B}(\mu_i)>0$, and therefore $M(1;\mu_i)>0$.

Therefore, for any $i\geqslant 0$, 0 is not an eigenvalue of the matrix $\mu_i\boldsymbol{I}-\boldsymbol{\Phi}_{\boldsymbol{w}}^{-1}(\tilde{\boldsymbol{w}})\boldsymbol{Z}_{\boldsymbol{w}}(\tilde{\boldsymbol{w}})$, and

$$\{i\geqslant 0\,|\,M(\mu_i)<0\}=\{1,2,\cdots,n\}.$$

Consequently,

$$\gamma_n:=\sum_{i\geqslant 0,\,M(\mu_i)<0}\dim E(\mu_i)=\sum_{i=1}^{n}\dim E(\mu_i),$$

which is odd by the hypothesis of the theorem. According to Lemma 5,

$$\operatorname{index}(\boldsymbol{G}(1;\cdot),\tilde{\boldsymbol{w}})=(-1)^{\gamma_n}=-1. \tag{30}$$

When $t=0$,

$$\begin{aligned}M(0;\mu)&=\det\{\tilde{\boldsymbol{\Phi}}_{\boldsymbol{w}}^{-1}(\tilde{\boldsymbol{w}})\}\det\{\mu\tilde{\boldsymbol{\Phi}}_{\boldsymbol{w}}(\tilde{\boldsymbol{w}})-\boldsymbol{Z}_{\boldsymbol{w}}(\tilde{\boldsymbol{w}})\}\\&=\det\{\tilde{\boldsymbol{\Phi}}_{\boldsymbol{w}}^{-1}(\tilde{\boldsymbol{w}})\}\tilde{\mathcal{B}}(\mu),\end{aligned} \tag{31}$$

where

$$\tilde{\mathcal{B}}(\mu)=\hat{d}_1\hat{d}_2\mu^2-\mu(\hat{d}_1a_{22}+\hat{d}_2a_{11})+(a_{11}a_{22}-a_{12}a_{21}).$$

Here $a_{11},a_{22},a_{12},a_{21}$ are given by (8).

By Theorem 7, the equation $\boldsymbol{G}(0;\boldsymbol{w})=0$ has only the constant positive solution $\tilde{\boldsymbol{w}}$ in $\boldsymbol{X}^+$. The condition $1-m<1/(a\alpha)$ in Theorem 8 implies $a_{11}<0$ and $\tilde{\mathcal{B}}(\mu_i)>0$ for all $i\geqslant 0$. Consequently, $M(0;\mu_i)=\det\{\boldsymbol{\Phi}_{\boldsymbol{w}}^{-1}(0;\tilde{\boldsymbol{w}})\}\cdot\tilde{\mathcal{B}}(\mu_i)>0$ for every $i\geqslant 0$. Thus, by Lemma 5,

$$\operatorname{index}(\boldsymbol{G}(0;\cdot),\tilde{\boldsymbol{w}})=(-1)^{\gamma}=(-1)^0=1. \tag{32}$$

According to Theorem 5 and Theorem 6, there exists a positive number ρ such that the positive solution of system (28) satisfies $\rho^{-1}<u,v<\rho$. Thus for any $t\in[0,1]$, $\boldsymbol{w}\in\partial H(\rho)$, $\boldsymbol{G}(t;\boldsymbol{w})\neq 0$. By homotopy invariance of the topological degree we get

$$\deg(\boldsymbol{G}(1;\cdot),0,H(\rho))=\deg(\boldsymbol{G}(0;\cdot),0,H(\rho)). \tag{33}$$

From (30) and (32),

$$\deg(\boldsymbol{G}(1;\cdot),0,H(\rho))=\operatorname{index}(\boldsymbol{G}(1;\cdot),\tilde{\boldsymbol{w}})=-1,$$

$$\deg(\boldsymbol{G}(0;\cdot),0,H(\rho))=\operatorname{index}(\boldsymbol{G}(0;\cdot),\tilde{\boldsymbol{w}})=1,$$

which contradicts (33).

3 Numerical Simulation

In this section, numerical simulations will be used in the one-dimensional case $\Omega=(0,2\pi)$ to verify the theoretical results obtained in the previous sections. Set the initial value condition $u_0=h_0(|\sin 4x|,|\sin 2x|)$, where h_0 is the initial value coefficient. The parameters in (5) are taken as

$$\begin{aligned}&a=8,\ r=1.5,\ b=25,\ c=4.3,\ m=0.64,\\&k=0.3,\ \alpha=0.3,\ \beta=0.1,\ d_1=0.01,\ d_2=1.\end{aligned} \tag{34}$$

It is easy to verify that they satisfy condition (H). Substituting (34) into equation (7) and the second equation of (6), we obtain $\tilde{u}\approx 4.284\,7$, $\tilde{v}\approx 0.642\,7$. So the constant positive solution of (4) and (5) is $\tilde{\boldsymbol{w}}=(\tilde{u},\tilde{v})\approx(4.284\,7,0.642\,7)$.

(i) Stability of the positive constant equilibrium $\tilde{\boldsymbol{w}}$ of (4) and (5).

Take the parameter values from (34). For the diffusion system (4), the initial value coefficients are taken as $h_0=0.2,0.4,0.8,1,1.2$. The numerical simulations are shown in Fig. 1. It can be seen that the positive constant equilibrium $\tilde{\boldsymbol{w}}$ of system (4) is locally asymptotically stable, which coincides with Theorem 3.

For the cross-diffusion system (5), take the parameters $d_{12}=100$, $d_{21}=1\,000$ and the initial value coefficient $h_0=0.1$. The numerical simulations are shown in Fig. 2, from which it can be seen that system (5) has a nonconstant positive solution. This result shows that cross-diffusion induces Turing instability of the positive constant equilibrium $\tilde{\boldsymbol{w}}$ of system (5), which is in agreement with Theorem 4.

(ii) Existence of nonconstant positive equilibrium of (5).

Take the parameters $d_{12}=100$, $d_{21}=1\,000$, $a=2$ and the initial value coefficient $h_0=0.1$. The rest of the parameters are taken as in (34). The numerical simulations are shown in Fig. 3, which reveals that the solution of (5) is time-independent. Thus, this solution represents a nonconstant positive equilibrium of (5), in agreement with Theorem 8.

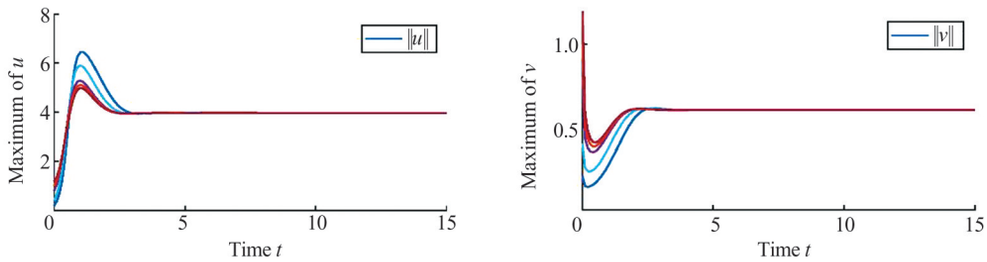


Fig. 1 Stability of the positive constant equilibrium $\tilde{w}$ of (4)

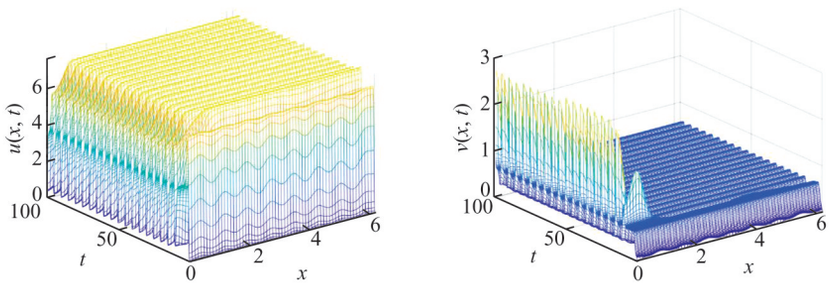


Fig. 2 Existence of the nonconstant positive solution of (5)

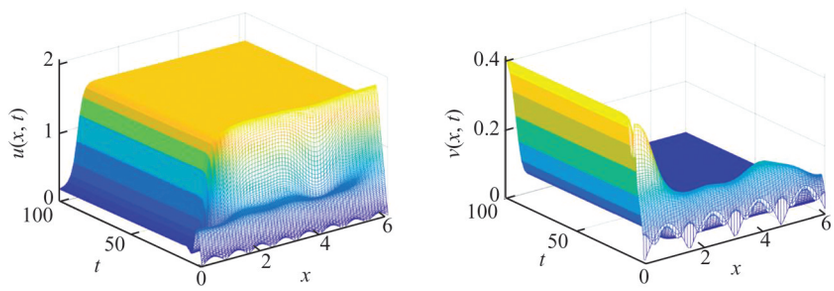


Fig. 3 Existence of the nonconstant positive equilibrium of (5)

(iii) The effect of the prey refuge proportion on the positive constant equilibrium.

Take the initial value coefficient $h_0 = 0.1$ and the parameter values from (34), and let m vary in [0, 1). Figure 4 shows the dependence of the positive constant equilibrium on m. As m increases, the value of the prey $\tilde{u}$ increases monotonically. In contrast, the value of the predator $\tilde{v}$ remains nearly constant for small m and then decreases when m exceeds a certain threshold. Biologically, this indicates that a higher refuge proportion benefits the prey by reducing predation pressure, allowing the prey population to grow. For the predator, a low level of prey refuge has little impact on its population size, but when the refuge proportion becomes sufficiently large, the reduced availability of prey leads to a decline in the predator population.

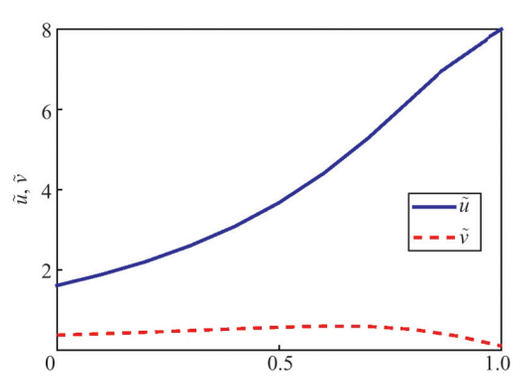


Fig. 4 Effect of m on $(\tilde{u}, \tilde{v})$

4 Conclusion

In this paper, we investigate the effect of cross-diffusion on positive solutions to a modified Leslie-Gower predator-prey model incorporating prey refuge and Crowley-Martin type functional response under homogeneous Neumann boundary conditions. The local asymptotic stability of the constant positive equilibrium is established using standard stability analysis. Conditions

for the onset of Turing instability are derived through linear stability analysis. The existence and nonexistence of nonconstant positive steady-state solutions are established by virtue of the Leray-Schauder degree theory, Young's inequality and Poincare's inequality. Finally, numerical simulations are carried out to verify the stability of the positive constant equilibrium. In the absence of cross-diffusion, the positive constant equilibrium remains stable, while it becomes unstable and induces nonconstant positive solutions when cross-diffusion is included. Furthermore, we verify the existence of nonconstant positive equilibria, and reveal the protective effect of prey refuge on prey populations as well as its shielding effect on predators. These results show that cross-diffusion is well suited for modeling interspecific and intraspecific interactions, as it can characterize the natural coexistence of predator-prey systems with non-uniform spatial distributions.

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交叉扩散驱动的具有避难所和Crowley-Martin功能反应的捕食-食饵模型的正解

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摘要:本文研究了在Neumann边界条件下, 一类具有食饵避难所和Crowley-Martin型功能反应函数的修正Leslie-Gower捕食-食饵模型。基于Routh-Hurwitz判据, 推导了无扩散模型和扩散模型唯一正常数稳态解的存在性与稳定性条件。此外, 研究了交叉扩散模型唯一正常数稳态解的Turing不稳定性。进而, 利用Young不等式、Poincaré不等式和Leray-Schauder度理论, 分别得到了扩散模型和交叉扩散模型非常数正稳态解的不存在性与存在性结果。最后, 通过若干组数值模拟, 展示了正常数稳态解的稳定性、Turing不稳定性、非常数稳态解的存在性以及食饵避难所的作用。结果表明, 交叉扩散是驱动模型中非常数正解(包括稳态解)产生的重要因素, 即交叉扩散可导致捕食者和被捕食者形成非均匀共存模式。

关键词:捕食-食饵模型; 交叉扩散; 图灵不稳定性; 非常数正稳态解; 数值模拟

□